

k -TUPLE RESTRAINED DOMINATION IN GRAPHS

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ABSTRACT. Let G be a graph of order n , and let $k \geq 1$ be an integer. A subset $S \subseteq V(G)$ of the vertices of G is a k -tuple restrained dominating set of G if each vertex in $V(G) - S$ is adjacent to at least k vertices in $V(G) - S$ and to at least k vertices in S , and each vertex in S is adjacent to at least $k - 1$ vertices in S . The minimum number of vertices of such a set in G is the k -tuple restrained domination number of G . Also, the maximum number of the classes of a partition of $V(G)$ such that its all classes are k -tuple restrained dominating sets of G is the k -tuple restrained domatic number of G .

In this paper, we present some sharp bounds for the k -tuple restrained domination number of a graph, and calculate it for some of the known graphs. Also, we mainly present basic properties of the k -tuple restrained domatic number of a graph.

1. Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [10]. Let $G = (V, E)$ be a graph with the *vertex set* V of order $n(G)$ and the *edge set* E of size $m(G)$. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$, while its cardinality is the *degree* of v . Similarly, the *open neighborhood* and the *closed neighborhood* of a subset $X \subseteq V(G)$ are $N_G(X) = \cup_{v \in X} N_G(v)$ and $N_G[X] = N_G(X) \cup X$, respectively. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If every vertex of G has degree k , then G is called *k -regular*. We write K_n , C_n and K_{n_1, \dots, n_p} for a *complete graph* or a *cycle* of order n , or a *complete p -partite graph*, respectively, while $G[X]$ denotes the *induced subgraph* of G by the vertex set X . The *complement* of a graph G , denoted \overline{G} , is a graph with the vertex set

MSC(2010): 05C69.

Keywords: k -tuple domination number, k -tuple domatic number, k -tuple restrained domination number, k -tuple restrained domatic number.

$V(G)$ and two disjoint vertices v and w are adjacent in \overline{G} if and only if they are not adjacent in G .

For each integer $k \geq 1$, the k -join $G \circ_k H$ of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H [7]. Also $G \circ_{*k} H$ denotes the k -join $G \circ_k H$ such that each vertex of G is joined to exactly k vertices of H .

Definition 1.1. [6] The *complementary prism* $G\overline{G}$ of a graph G is the graph formed from the disjoint union of G and \overline{G} by adding the edges of a perfect matching between the corresponding vertices (same label) of G and \overline{G} .

For example, the graph $C_5\overline{C_5}$ is the Petersen graph. Also, if $G = K_n$, the graph $K_n\overline{K_n}$ is the corona $K_n \circ K_1$, where the *corona* $G \circ K_1$ of a graph G is the graph obtained from G by attaching a pendant edge to each vertex of G .

Domination. The research of domination in graphs has been an evergreen of the graph theory. Its basic concept is the dominating set and the domination number. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [4, 5]. A numerical invariant of a graph which is in a certain sense dual to it is the domatic number of a graph. Many other variants of the dominating set were introduced and the corresponding numerical invariants were defined for them, which we here consider some of them from [3, 7, 8, 9].

Definition 1.2. [3] Let $k \geq 1$ be an integer and let G be a graph with $\delta(G) \geq k - 1$. A subset $S \subseteq V(G)$ is a k -tuple dominating set, briefly kDS, in G if for each $x \in V(G)$, $|N[x] \cap S| \geq k$. The minimum number of vertices of a kDS in G is the k -tuple domination number $\gamma_{\times k}(G)$ of G .

Definition 1.3. [7] Let $k \geq 1$ be an integer and let G be a graph with $\delta(G) \geq k$. A subset $S \subseteq V(G)$ is a k -tuple total dominating set, briefly kTDS, of G if for each $x \in V(G)$, $|N(x) \cap S| \geq k$. The minimum number of vertices of a kTDS of G is the k -tuple total domination number $\gamma_{\times k, t}(G)$ of G .

Definition 1.4. [8] In a graph $G = (V, E)$ with $\delta(G) \geq k \geq 1$, a k -tuple total restrained dominating set S , briefly kTRDS, of G is a k -tuple total dominating set of G such that each vertex of $V - S$ is adjacent to at least k vertices in $V - S$. The k -tuple total restrained domination number $\gamma_{\times k, t}^r(G)$ of G is the minimum cardinality of a kTRDS of G .

The domatic number $d(G)$ and the total domatic number $d_t(G)$ of a graph G were introduced in [1] and [2], respectively. Sheikholeslami and Volkmann in [9] extended the last definition to the k -tuple total domatic number as follows.

Definition 1.5. [9] The k -tuple total domatic partition, briefly kTDP, of G is a partition \mathcal{D} of the vertex set of G such that all classes of \mathcal{D} are k -tuple total dominating sets of G . The maximum number of classes of a k -tuple total domatic partition of G is the k -tuple total domatic number $d_{\times k,t}(G)$ of G .

In a similar way, Kazemi in [8] defined the k -tuple total restrained domatic number and the star k -tuple total restrained domatic number as follows.

Definition 1.6. [8] The k -tuple total restrained domatic partition, briefly kTRDP, of a graph G is a partition \mathcal{D} of the vertex set of G such that all classes of \mathcal{D} are k -tuple total restrained dominating sets in G . The maximum number of the classes of a kTRDP of G is the k -tuple total restrained domatic number $d_{\times k,t}^r(G)$ of G .

In this paper, we introduce and study two new concepts: k -tuple restrained domination number and k -tuple restrained domatic number.

Definition 1.7. In a graph G with $\delta(G) \geq k - 1$, a k -tuple restrained dominating set S , briefly kRDS, is a k -tuple dominating set of G such that each vertex of $V(G) - S$ is adjacent to at least k vertices of $V(G) - S$. The k -tuple restrained domination number $\gamma_{\times k}^r(G)$ of G is the minimum cardinality of a kRDS in G .

Definition 1.8. The k -tuple restrained domatic partition, briefly kRDP, of a graph G with $\delta(G) \geq k - 1$ is a partition of $V(G)$ to some k -tuple restrained dominating sets of G . The maximum number of the k -tuple restrained dominating sets of a kRDP of G is the k -tuple restrained domatic number $d_{\times k}^r(G)$ of G . Similarly, the star k -tuple restrained domatic number $d_{\times k}^{r*}(G)$ of G is the maximum number of the k -tuple restrained dominating sets of a kRDP of G such that at least one of them has cardinality $\gamma_{\times k}^r(G)$.

We recall that the 1-tuple restrained dominating set and the 1-tuple restrained domination number $\gamma_{\times 1}^r(G)$ of a graph G are known as the *restrained dominating set* and the *restrained domination number* $\gamma^r(G)$ of G , respectively.

Through this paper, k is a positive integer, and if v is a vertex in G , \bar{v} denotes its corresponding vertex in \overline{G} . Also for the cycle C_n of order n , we assume $V(C_n) = \{i \mid 1 \leq i \leq n\}$, and $E(C_n) = \{ij \mid 1 \leq i \leq n, \text{ and if } j \equiv i + 1 \pmod{n}\}$.

This paper is organized as follows. In Section 2, we calculate $\gamma_{\times k}^r(G)$, where G is K_n , C_n , $\overline{C_n}$ or a bipartite graph. Also we present some bounds for the k -tuple restrained domination number of the complete multipartite graph. Then, in Section 3, we characterize the structure of graphs G that $\gamma_{\times k}^r(G) =$

m , for each $m \geq k$, and give upper and lower bounds for $\gamma_{\times k}^r(G)$ when G is an arbitrary graph. In Section 4, we mainly present basic properties of the k -tuple restrained domatic number of a graph and give some bounds for it. Also we give some sufficient conditions for that the k -tuple domination (respectively domatic) number of a graph be equal to the k -tuple restrained domination (respectively domatic) number of it. Finally, in the last section, we give some sharp bounds for the k -tuple restrained domination number of the complementary prism $G\overline{G}$ in terms of the same numbers of G and \overline{G} .

The following proposition is useful in our investigation.

Proposition 1.9. (Kazemi [8] 2011) *For any graph G with $\delta(G) \geq k \geq 1$, $d_{\times k, t}^r(G) = d_{\times k, t}(G)$.*

2. k -tuple restrained domination in some graphs

In this section, we calculate $\gamma_{\times k}^r(G)$, where G is K_n , C_n , $\overline{C_n}$ or a bipartite graph. Also we present some bounds for the k -tuple restrained domination number of a complete multipartite graph. First we state the following observation.

Observation 2.1. *For any graph G of order n ,*

- i. $d_{\times k, t}^r(G) \leq d_{\times k}^r(G) \leq d_{\times k}(G)$,
- ii. $d_{\times k, t}^r(G) \leq d_{\times k, t}(G) \leq d_{\times k}(G)$,
- iii. $d_{\times k}^r(G) = 1$ if $\delta(G) \leq 2k - 1$,
- iv. $\Delta(G) \geq 2k$ and $\gamma_{\times k}^r(G) \leq n - k - 1$ if $\gamma_{\times k}^r(G) < n$, and so $n \geq 2k + 1$,
- v. every vertex of degree at most $2k - 1$ of G and at least $k - 1$ of its neighbors belong to every k RDS.

Proposition 2.2. *For any integers $n > k \geq 1$,*

$$\gamma_{\times k}^r(K_n) = \begin{cases} n & \text{if } n \leq 2k, \\ k & \text{otherwise.} \end{cases}$$

Proof. Observation 2.1 (v) implies that $\gamma_{\times k}^r(K_n) = n$ if $n \leq 2k$. Since also every k -subset of vertices is a k RDS in K_n , we obtain $\gamma_{\times k}^r(K_n) = k$ if $n \geq 2k + 1$. \square

In the next three propositions, we find the k -tuple restrained domination number of a cycle and its complement.

Proposition 2.3. *For any integer $n \geq 3$,*

$$\gamma_{\times 2}^r(C_n) = \gamma_{\times 3}^r(C_n) = n,$$

and

$$\gamma^r(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2 \pmod{3}, \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof. Observation 2.1 (v) implies that $\gamma_{\times 2}^r(C_n) = \gamma_{\times 3}^r(C_n) = n$. Now let $n \equiv r \pmod{3}$, where $0 \leq r \leq 2$. Since the sets $S_0 = \{1+3i \mid 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\}$, $S_1 = S_0 \cup \{n\}$ and $S_2 = S_1 \cup \{n-1\}$ are restrained dominating sets in C_n with smallest cardinality, when r is 0, 1, 2 respectively, we obtain

$$\gamma^r(C_n) = \begin{cases} \lfloor \frac{n}{3} \rfloor + 1 & \text{if } n \equiv 2 \pmod{3}, \\ \lfloor \frac{n}{3} \rfloor & \text{otherwise.} \end{cases}$$

□

Proposition 2.4. *Let $n \geq 4$. Then*

$$\gamma^r(\overline{C_n}) = \begin{cases} 4 & \text{if } n = 4, \\ 3 & \text{if } n = 5, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If $n = 4$, then $\overline{C_4} = 2K_2$, and so $\gamma^r(\overline{C_4}) = 2\gamma^r(\overline{K_2}) = 4$, by Proposition 2.2. Let $n \geq 5$. Then $\gamma^r(\overline{C_n}) \geq 2$. It can easily be verified that $\gamma^r(\overline{C_5}) = 3$. Since $\{\overline{1}, \overline{4}\}$ is a RDS of $\overline{C_n}$ for $n \geq 6$, we obtain $\gamma^r(\overline{C_n}) = 2$. □

Proposition 2.2 implies that $\gamma_{\times 2}^r(\overline{C_4}) = 4$. In the next proposition we find $\gamma_{\times k}^r(\overline{C_n})$, when $n \geq 5$ and $k \geq 2$.

Proposition 2.5. *Let $n \geq 5$ and let $2 \leq k \leq n - 3$. Then*

$$\gamma_{\times k}^r(\overline{C_n}) = \begin{cases} n & \text{if } n \leq 2k + 2, \\ k + 2 & \text{if } 2k + 3 \leq n \leq 3k, \\ k + 1 & \text{otherwise.} \end{cases}$$

Proof. Note that $\overline{C_n}$ is $(n-3)$ -regular. If $n \leq 2k + 2$, then $\gamma^r(\overline{C_n}) = n$, by Observation 2.1 (v). Let $n \geq 2k + 3$. Then obviously $\gamma_{\times k}^r(\overline{C_n}) \geq k + 1$. Let S be an arbitrary k RDS in $\overline{C_n}$. It can easily be verified that $|S| = k + 1$ if and only if there exist at most two vertices \bar{i} and \bar{j} in S such that $|j - i| = 1$, and for the other different vertices \bar{t} and $\bar{\ell}$, $|\ell - t| \geq 3$. Since this holds if and only if $n \geq 3k + 1$, we obtain $\gamma_{\times k}^r(\overline{C_n}) \geq k + 2$ if $2k + 3 \leq n \leq 3k$. Now by considering the sets $S = \{1\} \cup \{\overline{3i+2} \mid 0 \leq i \leq k\}$ and $S' = \{\overline{2i+1} \mid 0 \leq i \leq k+1\}$ as k -tuple restrained dominating sets of $\overline{C_n}$ for $n \geq 3k + 1$ and $2k + 3 \leq n \leq 3k$, respectively, we obtain

$$\gamma_{\times k}^r(\overline{C_n}) = \begin{cases} k + 2 & \text{if } 2k + 3 \leq n \leq 3k, \\ k + 1 & \text{otherwise.} \end{cases}$$

□

Now we present some bounds on the k -tuple restrained domination number of a bipartite graphs.

Proposition 2.6. *Let G be a bipartite graph with $\delta(G) \geq k - 1 \geq 1$. Then*

$$2k - 2 \leq \gamma_{\times k}^r(G) \leq n.$$

Further, $\gamma_{\times k}^r(G) = 2k - 2$ if and only if G is isomorphic to the complete bipartite graph $K_{k-1, k-1}$.

Proof. Let G be a bipartite graph with $\delta(G) \geq k - 1 \geq 1$, which is partitioned to the independent sets X and Y . Let S be an arbitrary kRDS of G , and let $w \in X$ and $z \in Y$. The definition of k -tuple restrained dominating set implies that $|S \cap N(w)| \geq k - 1$ and $|D \cap N(z)| \geq k - 1$. Since $N(w) \cap N(z) = \emptyset$, we deduce that $|D| \geq 2k - 2$ and thus $2k - 2 \leq \gamma_{\times k, t}^r(G) \leq n$.

If G is isomorphic to $K_{k-1, k-1}$, then $\gamma_{\times k}^r(G) = 2k - 2$, by Observation 2.1 (v). Now let S be a $\gamma_{\times k}^r$ -set of G of cardinality $2k - 2$. Then $|S \cap X| = |S \cap Y| = k - 1$. Since every vertex in X or Y is adjacent to exactly $k - 1$ vertices in S , we conclude that $X - S = Y - S = \emptyset$. Therefore, G is isomorphic to the complete bipartite graph $K_{k-1, k-1}$. \square

Corollary 2.7. *Let G be a bipartite graph with $\delta(G) \geq k - 1 \geq 1$. If G is not isomorphic to $K_{k-1, k-1}$, then $2k \leq \gamma_{\times k}^r(G) \leq n$.*

Corollary 2.8. *For any integers $n \geq m \geq k - 1 \geq 1$,*

$$\gamma_{\times k}^r(K_{n, m}) = \begin{cases} 2k & \text{if } n \geq m \geq 2k, \\ k + m & \text{if } n \geq 2k > m, \\ n + m & \text{if } 2k > n \geq m \geq k - 1. \end{cases}$$

Now we present some bounds for the k -tuple restrained domination number of a complete p -partite graph, when $p \geq 3$.

Proposition 2.9. *For any integer $p \geq 3$, if G is a complete p -partite graph, then*

$$\gamma_{\times k}^r(G) \geq \lceil \frac{p(k-1)}{p-1} \rceil.$$

Proof. Let $G = K_{n_1, \dots, n_p}$ be a complete p -partite graph with the partition $V(G) = X_1 \cup \dots \cup X_p$ to the independent sets X_1, \dots, X_p , where $|X_i| = n_i$ for each i . Let $n = n_1 + \dots + n_p$, and let S be an arbitrary kRDS of G , in which $S_i = X_i \cap S$ and $|S_i| = s_i$ for each i . Since every vertex in X_i is adjacent to at least $k - 1$ vertices in $S - X_i = \bigcup_{j=1, j \neq i}^p S_j$, we obtain

$$\sum_{j=1}^p s_j - s_i \geq k - 1,$$

for each $1 \leq i \leq p$. Hence $(p - 1)|S| \geq p(k - 1)$, and so $|S| \geq \lceil \frac{p(k-1)}{p-1} \rceil$. Since S is an arbitrary kRDS of G , we obtain $\gamma_{\times k}^r(G) \geq \lceil \frac{p(k-1)}{p-1} \rceil$. \square

For giving an upper bound for the k -tuple restrained domination number of a complete p -partite graph G with $p \geq 3$, we use the following definitions and notations.

Let $G = K_{n_1, \dots, n_p}$ be a complete p -partite graph with the partition $V(G) = X_1 \cup \dots \cup X_p$ to the independent sets X_1, \dots, X_p , where $|X_i| = n_i$ for each i . Let $n = n_1 + \dots + n_p$, and let S be a kRDS of G and let $S_i = X_i \cap S$, $S'_i = X_i - S$ and $|S_i| = s_i$ for each i . Consider $t(S)$ as the number of indices i that $s_i < n_i$ and define

$$t_0 = \min\{t(S) | S \text{ is a kRDS in } G\}.$$

Then $t(S) \geq 1$, because $t_0 = 0$ if and only if $\gamma_{\times k}^r(G) = n$. Obviously $t(S) \geq 2$. Without loss of generality, we may assume that $s_i < n_i$ if and only if $1 \leq i \leq t(S)$. Let $w_j \in X_j - S = X_j - S_j$ for each $1 \leq j \leq t(S)$. Since S is a kRDS of G , we obtain $|N(w_j) \cap (V - S)| \geq k$. Hence for any $1 \leq j \leq t(S)$,

$$\begin{aligned} k &\leq |N(w_j) \cap (V - S)| \\ &= \sum_{i=1, i \neq j}^{t(S)} |N(w_j) \cap S'_i| \\ &= \sum_{i=1, i \neq j}^{t(S)} |S'_i| \\ &= \sum_{i=1}^{t(S)} |S'_i| - |S'_j|. \end{aligned}$$

By summing the inequalities, we obtain

$$\begin{aligned} t(S)k &\leq (t(S) - 1) \sum_{i=1}^{t(S)} (n_i - s_i) \\ &= (t(S) - 1) \sum_{i=1}^p (n_i - s_i) \\ &= (t(S) - 1)(n - |S|). \end{aligned}$$

Hence $|S| \leq n - k - \lceil \frac{k}{t(S)-1} \rceil$. Since S was arbitrary, we obtain

$$\gamma_{\times k}^r(G) \leq n - k - \lceil \frac{k}{t_0 - 1} \rceil.$$

Therefore, we have proved the next proposition.

Proposition 2.10. *For any integer $p \geq 3$, let G be a complete p -partite graph of order n . If $\gamma_{\times k}^r(G) < n$, then $\gamma_{\times k}^r(G) \leq n - k - \lceil \frac{k}{t_0 - 1} \rceil$.*

3. Some bounds

In this section, for any integer $m \geq k$, we characterize graphs G with $\gamma_{\times k, t}^r(G) = m$. Then we present some bounds for $\gamma_{\times k}^r(G)$ in terms on k , the order and the size of G .

Theorem 3.1. *Let G be a graph with $\delta(G) \geq k - 1 \geq 1$. Then for any integer $m \geq k$, $\gamma_{\times k}^r(G) = m$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$, for some graph F and some spanning subgraph K'_m of K_m with $\delta(F) \geq k$ and $\delta(K'_m) \geq k - 1$ and m is minimum with this property.*

Proof. For any integers $m \geq k \geq 2$, let S be a $\gamma_{\times k}^r(G)$ -set of cardinality m . Then every vertex v has at least k or $k - 1$ neighbors in S if $v \in V(G) - S$ or $v \in S$, respectively. Also, every vertex in $V(G) - S$ has at least k neighbors in $V(G) - S$. Thus $G[S] = K'_m$, for some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k - 1$. If $|V(G)| = m$, then $G = K'_m$. If $|V(G)| > m$, let F be the induced subgraph $G[V(G) - S]$. Then $\delta(F) \geq k$ and $G = F \circ_k K'_m$. Also the definition of the k -tuple restrained domination number implies that m is minimum with this property.

Conversely, let $G = K'_m$ or $G = F \circ_k K'_m$, for some graph F with $\delta(F) \geq k$ and some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k - 1$ such that m is minimum with this property. Then $\gamma_{\times k}^r(G) \leq m$, since $V(K'_m)$ is a kRDS of G of cardinality m . If $\gamma_{\times k}^r(G) = m' < m$, then the previous paragraph implies that for some graph F' with $\delta(F') \geq k$ and some spanning subgraph $K'_{m'}$ of $K_{m'}$ with $\delta(K'_{m'}) \geq k - 1$, $G = F' \circ_k K'_{m'}$, that contradicts this fact that m is minimum with this property. Therefore $\gamma_{\times k}^r(G) = m$. \square

Corollary 3.2. *Let G be a graph with $\delta(G) \geq k - 1 \geq 1$. Then $\gamma_{\times k}^r(G) = k$ if and only if $G = K_k$ or $G = F \circ_k K_k$, for some graph F with $\delta(F) \geq k$.*

Theorem 3.3. *If G is a graph with minimum degree at least $k - 1$ on n vertices and with m edges, then*

$$(3.1) \quad \gamma_{\times k}^r(G) \geq \frac{3kn - 2m}{2k + 1},$$

*with equality if and only if there exist a $(k - 1)$ -regular graph H of order $\gamma_{\times k}^r(G)$ and a k -regular graph F of order $n - \gamma_{\times k}^r(G)$ such that G is isomorphic to $F \circ_{*k} H$.*

Proof. Let $G = (V, E)$ be a graph with minimum degree at least $k - 1$ on n vertices and with m edges, and let S be a kRDS of G with minimum cardinality. Since S is also a kDS in G , $\delta(G[S]) \geq k - 1$ and $\delta(G[V - S]) \geq k$, we obtain

the following inequalities:

$$\begin{aligned} m_1 &\geq \frac{(k-1)\gamma_{\times k}^r(G)}{2}, \\ m_2 &\geq \frac{k(n-\gamma_{\times k}^r(G))}{2}, \\ m_3 &\geq k(n-\gamma_{\times k}^r(G)), \end{aligned}$$

where m_1 and m_2 are respectively the number of edges in the induced subgraphs $G[S]$ and $G[V-S]$ and m_3 is the number of edges connecting vertices of $V-S$ to the vertices of S . By summing the inequalities, we obtain

$$m = m_1 + m_2 + m_3 \geq \frac{3kn}{2} - \frac{2k+1}{2}\gamma_{\times k}^r(G),$$

and thus $\gamma_{\times k}^r(G) \geq \frac{3kn-2m}{2k+1}$.

We know that equality holds in (3.1) if and only if the inequality occurring in the proof becomes equality, that is,

$$\begin{aligned} m_1 &= \frac{(k-1)\gamma_{\times k}^r(G)}{2}, \\ m_2 &= \frac{k(n-\gamma_{\times k}^r(G))}{2}, \\ m_3 &= k(n-\gamma_{\times k}^r(G)). \end{aligned}$$

The first and second equalities mean that there exist the $(k-1)$ -regular graph $H = G[S]$ of order $\gamma_{\times k}^r(G)$ and the k -regular graph $F = G[V(G)-S]$ of order $n - \gamma_{\times k}^r(G)$, respectively, while the third equality means that every vertex of F is adjacent to exactly k vertices of H . Hence equality holds in (3.1) if and only if G is isomorphic to $F \circ_{*k} H$. \square

As an example, if G is the complete graph K_{2k+1} , then $\gamma_{\times k}^r(G) = \frac{3kn-2m}{2k+1}$.

Corollary 3.4. *For any graph G with n vertices and m edges, $\gamma^r(G) \geq n - \frac{2m}{3}$.*

4. Basic properties

In this section, we mainly present basic properties of the k -tuple restrained domatic number of a graph and give some other bounds on it. First, we give a proposition which its proof is left to the reader.

Proposition 4.1. *For any integers $n \geq k \geq 1$, $d_{\times k}^r(K_n) = \lfloor \frac{n}{k} \rfloor$.*

Theorem 4.2. *If G is a graph of order n with $\delta(G) \geq k-1$, then*

$$\gamma_{\times k}^r(G) \cdot d_{\times k}^r(G) \leq n.$$

Moreover, if $\gamma_{\times k}^r(G) \cdot d_{\times k}^r(G) = n$, then for each k RD $\{V_1, V_2, \dots, V_d\}$ of G with $d = d_{\times k}^r(G)$, each set V_i is a $\gamma_{\times k}^r(G)$ -set.

Proof. Let $\{V_1, V_2, \dots, V_d\}$ be a kRDP of G such that $d = d_{\times k}^r(G)$. Then

$$\begin{aligned} d \cdot \gamma_{\times k}^r(G) &= \sum_{i=1}^d \gamma_{\times k}^r(G) \\ &\leq \sum_{i=1}^d |V_i| \\ &= n. \end{aligned}$$

If $\gamma_{\times k}^r(G) \cdot d_{\times k}^r(G) = n$, then the inequality occurring in the proof becomes equality. Hence for the kRDP $\{V_1, V_2, \dots, V_d\}$ of G and for each i , $|V_i| = \gamma_{\times k}^r(G)$. Thus each set V_i is a $\gamma_{\times k}^r(G)$ -set. \square

An immediate consequence of Theorem 4.2 and Corollary 3.2 now follows.

Corollary 4.3. *If G is a graph of order n with $\delta(G) \geq k - 1$, then*

$$d_{\times k}^r(G) \leq \frac{n}{k},$$

with equality if and only if $G = K_k$ or $G = F \circ_k K_k$ for some graph F with $\delta(F) \geq k$.

Proposition 2.6 and Corollary 2.7 improve the bound given in Corollary 4.3 for bipartite graphs.

Proposition 4.4. *Let G be a bipartite graph of order n with $\delta(G) \geq k - 1 \geq 1$. If G is isomorphic to the complete bipartite graph $K_{k-1, k-1}$, then $d_{\times k}^r(G) = \frac{n}{2k-2}$, and if G is not isomorphic to $K_{k-1, k-1}$, then $d_{\times k}^r(G) \leq \frac{n}{2k}$.*

The next theorem presents a sufficient condition for the k -tuple restrained domatic number of a graph be equal to the its k -tuple domatic number.

Theorem 4.5. *Let G be a graph with $\delta(G) \geq k - 1 \geq 0$. If $(d_{\times k}(G), d_{\times k, t}(G)) \neq (2, 1)$, then $d_{\times k}^r(G) = d_{\times k}(G)$.*

Proof. If $d_{\times k}(G) = 1$, then $d_{\times k}^r(G) = d_{\times k}(G)$, by Observation 2.1 (i). Now let $d_{\times k}(G) \geq 3$. Let $d = d_{\times k}(G)$ and let $\mathcal{D} = \{D_1, \dots, D_d\}$ be a k -tuple domatic partition of G . Choose D_1 as an arbitrary class of \mathcal{D} . Let $x \in V(G) - D_1 = D_2 \cup D_3 \cup \dots \cup D_d$. Without loss of generality, let $x \in D_2$. As D_1 is a kDS of G , there exists a k -set S_x^1 such that $S_x^1 \subseteq N(x) \cap D_1$. Also since D_3 is a kDS of G , there exists a k -set S_x^3 such that $S_x^3 \subseteq N(x) \cap D_3$. We have $S_x^3 \subseteq V(G) - D_1$, because $D_1 \cap D_3 = \emptyset$. On the other hand, if $x \in D_1$, then $|N[x] \cap D_1| \geq k$, because D_1 is a kDS of G . Therefore, we have proved that D_1 is a kRDS of G . The set D_1 was chosen arbitrarily, therefore \mathcal{D} is a k -tuple restrained domatic partition of G and $d_{\times k}(G) \leq d_{\times k}^r(G)$. Hence $d_{\times k}(G) = d_{\times k}^r(G)$, by Observation 2.1 (i).

Now let $d_{\times k}(G) = d_{\times k,t}(G) = 2$. Then $d_{\times k,t}^r(G) = 2$, by Proposition 1.9, and so

$$2 = d_{\times k,t}^r(G) \leq d_{\times k}^r(G) \leq d_{\times k}(G) = 2,$$

by Observation 2.1 (i), which implies $d_{\times k}(G) = d_{\times k}^r(G)$. \square

Corollary 4.6. *Let G be a graph. If $(d(G), d_t(G)) \neq (2, 1)$, then $d^r(G) = d(G)$.*

The condition $(d_{\times k}(G), d_{\times k,t}(G)) \neq (2, 1)$ in Theorem 4.5 is necessary. For example, if $G = K_{2k+1}$, then $d_{\times k}(G) = 2$, $d_{\times k,t}(G) = 1$ but $d_{\times k}^r(G) = 1$. In continuation, we present a sufficient condition for $\gamma_{\times k}^r(G) = \gamma_{\times k}(G)$.

Theorem 4.7. *Let G be a graph with $\delta(G) \geq k - 1 \geq 1$. If $d_{\times k}^*(G) \geq 3$, then $\gamma_{\times k}^r(G) = \gamma_{\times k}(G)$.*

Proof. Since every k RDS of G is also a k DS of G , we have $\gamma_{\times k}(G) \leq \gamma_{\times k}^r(G)$. For the converse inequality, since $d_{\times k}^*(G) \geq 3$, let S , S' and S'' be three pairwise disjoint k -tuple dominating sets of G such that S is a $\gamma_{\times k}(G)$ -set. Since S' and S'' are also two k -tuple dominating sets of G , each vertex $x \in V(G) - S$ is adjacent to at least $k - 1$ vertices in S' and to at least $k - 1$ vertices in S'' . Also this facts that S' and S'' are disjoint and $k \geq 2$ imply that x is adjacent to at least k vertices in $V(G) - S$. Hence S is a k RDS of G , and $\gamma_{\times k}^r(G) \leq |S| = \gamma_{\times k}(G)$. The previous two inequalities imply $\gamma_{\times k}^r(G) = \gamma_{\times k}(G)$. \square

The converse of Theorem 4.7 does not hold. For example, if $5 \leq 2k + 1 \leq n \leq 3k - 1$, then $\gamma_{\times k}^r(K_n) = \gamma_{\times k}(K_n) = k$ but $d_{\times k}^*(K_n) = 2$. Also the following example shows that the condition $d_{\times k}^*(G) \geq 3$ in Theorem 4.7 can not be replaced by $d_{\times k}(G) \geq 3$.

Example 4.8.

Let G be a graph of order 16 with $V(G) = \{1, 2, 3, \dots, 16\}$ and

$$\begin{aligned} E(G) = & \{(1, 2), (1, 3), (1, 4), (2, 5), (2, 6), (2, 7), (2, 8), \\ & (3, 9), (3, 10), (3, 11), (3, 12), (4, 13), (4, 14), (4, 15), \\ & (4, 16), (5, 6), (7, 8), (9, 10), (11, 12), (13, 14), (15, 16)\}. \end{aligned}$$

Then $\gamma(G) = 3$, because the set $\{2, 3, 4\}$ is the unique dominating set of smallest cardinality. Also $\gamma^r(G) = 4$. For example the set $\{3, 4, 5, 6\}$ is a restrained dominating set in smallest cardinality. Since each of the sets $\{1, 2, 9, 11, 13, 15\}$, $\{3, 5, 7, 14, 16\}$ and $\{4, 6, 8, 10, 12\}$ are three disjoint dominating sets of G , then $d(G) \geq 3$.

5. The complementary prisms

First, we calculate the k -tuple restrained domination number of the complementary prism of a regular graph for some integer k .

Theorem 5.1. *Let k and ℓ be integers such that $0 \leq k - 2 \leq \ell \leq 2k - 3$. If G is a ℓ -regular graph of order n , then*

$$\gamma_{\times k}^r(G\overline{G}) \geq n + k,$$

with equality if and only if $n \geq \ell + 2k$ and $V(\overline{G})$ contains a k -subset T such that for each vertex $\bar{i} \in V(\overline{G})$, $|N(\bar{i}) \cap T| \geq k - 1$ and $|N(\bar{i}) \cap (V(\overline{G}) - T)| \geq k$ if $\bar{i} \in V(\overline{G}) - T$, and $|N(\bar{i}) \cap T| \geq k - 2$ otherwise.

Proof. Let $V(G\overline{G}) = V(G) \cup V(\overline{G})$ such that $V(G) = \{i | 1 \leq i \leq n\}$ and $V(\overline{G}) = \{\bar{i} | 1 \leq i \leq n\}$. Let $n \geq 2k + \ell$, and let S be an arbitrary k RDS of $G\overline{G}$. Since each vertex i has degree $\ell + 1 \leq 2k - 2$, Observation 2.1 (v) implies $V(G) \subseteq S$. Let $\bar{i} \notin S$. Then $|N(\bar{i}) \cap V(\overline{G}) \cap S| \geq k - 1$. If $|N(\bar{i}) \cap V(\overline{G}) \cap S| \geq k$, then we have nothing to prove. Thus let $N(\bar{i}) \cap V(\overline{G}) \cap S = \{\bar{j}_i | 1 \leq i \leq k - 1\}$. This implies that there exists a vertex $\bar{t} \in S - \{\bar{j}_i | 1 \leq i \leq k - 1\}$ such that its corresponding vertex t in G is adjacent to some vertex j_i , when $1 \leq i \leq k - 1$. Hence $|S| \geq n + k$. Since S was arbitrary, we obtain $\gamma_{\times k}^r(G\overline{G}) \geq n + k$.

It can easily be verified by the reader that $\gamma_{\times k}^r(G\overline{G}) = n + k$ if and only if $n \geq \ell + 2k$ and $V(\overline{G})$ contains a k -subset T such that for each vertex $\bar{i} \in V(\overline{G})$, $|N(\bar{i}) \cap T| \geq k - 1$ and $|N(\bar{i}) \cap (V(\overline{G}) - T)| \geq k$ if $\bar{i} \in V(\overline{G}) - T$, and $|N(\bar{i}) \cap T| \geq k - 2$ otherwise. \square

Corollary 5.2. *Let k and ℓ be integers such that $0 \leq k - 2 \leq \ell \leq 2k - 3$. If G is a ℓ -regular graph of order $n \leq \ell + 2k - 1$, then $\gamma_{\times k}^r(G\overline{G}) = 2n$.*

Corollary 5.3. *For any integer $n \geq 4$,*

$$\gamma_{\times 2}^r(C_n \overline{C_n}) = \begin{cases} 2n & \text{if } n = 4, 5, \\ n + 2 & \text{otherwise.} \end{cases}$$

Corollary 5.4. *For any integer $n \geq 5$,*

$$\gamma_{\times 3}^r(C_n \overline{C_n}) = \begin{cases} 2n & \text{if } n = 5, 6, 7, \\ n + 3 & \text{otherwise.} \end{cases}$$

The next theorem gives some lower and upper bounds for $\gamma_{\times k}^r(G\overline{G})$, where G is an arbitrary graph.

Theorem 5.5. *If G is a graph of order n with $\min\{\delta(G), \delta(\overline{G})\} \geq k - 1$, then*

$$\gamma_{\times(k-1)}^r(G) + \gamma_{\times(k-1)}^r(\overline{G}) \leq \gamma_{\times k}^r(G\overline{G}) \leq \gamma_{\times k}^r(G) + \gamma_{\times k}^r(\overline{G}),$$

where $k \geq 2$ in the lower bound and $k \geq 1$ in the upper bound.

Proof. To prove the lower bound, let $k \geq 2$ and let D be a k RDS of $G\overline{G}$. Since every vertex in $V(G)$ (respectively in $V(\overline{G})$) is adjacent to only one vertex in $V(\overline{G})$ (respectively in $V(G)$), we have a nontrivial partition $D = D' \cup D''$ such that D' is a $(k-1)$ RDS of G and D'' is a $(k-1)$ RDS of \overline{G} . Hence

$$\gamma_{\times(k-1)}^r(G) + \gamma_{\times(k-1)}^r(\overline{G}) \leq |D'| + |D''| = |D| = \gamma_{\times k}^r(G\overline{G}).$$

To prove the upper bound, let $k \geq 1$ and let S and S' be a $\gamma_{\times k}^r(G)$ -set and a $\gamma_{\times k}^r(\overline{G})$ -set, respectively. Then the set $S \cup S'$ is a k RDS of $G\overline{G}$, implying that

$$\gamma_{\times k}^r(G\overline{G}) \leq \gamma_{\times k}^r(G) + \gamma_{\times k}^r(\overline{G}).$$

□

Propositions 2.3, 2.4, 2.5 and Corollaries 5.3, 5.4 shows that if $G = C_n$, then the lower bound in Theorem 5.5 is sharp for $k = 3$ and $5 \leq n \neq 7$, and also the upper bound in Theorem 5.5 is sharp for $k = 3$ and $5 \leq n \leq 7$, or $(k, n) = (2, 5)$.

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